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Analytical Relationships for Linear Quadratic Aeroelastic Flight Control Eigenvalues

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Several obstacles limiting the use of contemporary control design techniques in production vehicles are noted. These obstacles restrict one from obtaining insight into the control law augmentation of the vehicle. For example, with the linear quadratic (LQ) state feedback method, detail effects from cost function weight adjustment are not easily understood. Further, the role of each feedback loop is not readily apparent. Fundamental to overcoming this problem is factoring of the LQ characteristic polynomial. Useful analytical expressions, and ultimately the relationships they represent, are sought between basic design parameters of the system and the closed-loop eigenvalues. An approximate analytical factoring technique, previously developed for open-loop applications, is considered as a tool for this closed-loop problem. The first two terms of a Taylor series are used to capture the polynomial coefficient dependencies on the polynomial factors. By analytically inverting the first-order sensitivity matrix, corrections to preliminary approximate factors are generated. Expressions for the closed-loop factors are in terms of basic parameters such as stability and control derivatives, structural vibration damping ratio and natural frequency, and cost function weights, allowing key relationships to be uncovered between the design knobs and important closed-loop features.

Introduction

INCREASED reliance on flight control systems to meet mission design requirements while retaining exceptional flying qualities has been the trend for several decades.[1] In advanced vehicle concepts, where overall flight characteristics are influenced by various factors such as rigid-body motion, structural vibration, unsteady aerodynamic flow, and propulsion system behavior, multiple feedback loops are designed to actively control, in an integrated fashion, key dynamic features in the vehicle system. Often these control loops must operate near maximum performance levels with stability margins approaching minimum requirement limits for economic or functional viability of the concept.

The theoretical control community has been concentrating on contemporary design techniques for such applications for over three decades. Powerful methods such as linear quadratic regulator/linear quadratic Gaussian/loop transfer recovery (LTR), H_2, and H_∞ have been created to close multiple feedback loops simultaneously in a well-coordinated manner.[2,3] Recent activity has concentrated on tuning existing methods to provide closed-loop robustness properties. Two obstacles limiting the use of these methods in production vehicles are 1) the lumped nature of the algorithms masking individual loop significance and 2) theoretical–mathematical sophistication linking relevant closed-loop properties to the design knobs. These characteristics limit insight into the control law's stabilization and/or augmentation of the vehicle, and if design results suggest the need for modifications, the flight control designer is often unclear as to how to proceed, short of quasi-iterative strategies.

For example, with standard linear quadratic (LQ) state feedback control, once the input–output suite is specified, the designer must select cost function weights that provide acceptable trades between stability and performance, in both time- and frequency-domain measures. Effects from adjusting a particular weight, as it passes through the Riccati equation and into a specific feedback gain or closed-loop characteristic, is not easily understood. Further, the role each feedback loop plays in stabilizing/augmenting specific plant modes is not readily apparent, as compared with a conventional, sequential loop closure strategy.[4] The overall objective of this work is to initiate a small step toward bridging this gap between theory and application for specific flight control problems.

LQ-based root locus theory provides an efficient origin for pursuing this activity. It is well understood how the closed-loop characteristic polynomial is explicitly defined in terms of the plant transfer function numerator and denominator polynomials and cost weightings. Therefore, analytical factoring of the closed-loop characteristic polynomial is fundamental to fostering useful relationships between the design knobs and resulting closed-loop features.[5] For single-loop systems, this relationship is portrayed graphically as the cost weight ratio is varied.[2] For multiloop systems, this graphical information is quite sparse, existing only in an asymptotic sense.[6–9]

Exact symbolic factoring of polynomials is only possible for polynomials of degree four or less. Noting the symmetric nature of the LQ characteristic polynomial, exact analysis is restricted to dynamic systems of order four or less. Further, the cubic and quartic formulas are not easily used for problem understanding. Therefore, attention is turned to an approximate analytical factoring technique that has been used successfully to develop relationships for open-loop elastic airframe transfer function poles and zeros in terms of basic parameters such as stability and control derivatives and structural vibration parameters such as damping ratio and natural frequency.[10–15] In this technique, the first two terms of a Taylor series are used to capture the polynomial coefficient dependencies on the polynomial factors. By analytically inverting the first-order sensitivity matrix, corrections to preliminary approximate factors can be generated. In principle, the technique is applicable to higher-order systems, but computational complexity restricts the upper limit on system dynamic order that can be handled, in practice. Relatively simple and accurate analytical expressions conducive for obtaining insight into the vehicle physics have been obtained by this technique.

The strategy here is to explore the utilization of this factoring tool in closed-loop settings, and to generate analytical expressions, albeit approximate, between the cost function weights and closed-loop eigenvalues, for a LQ-based aeroelastic flight control law. Through these expressions, the goal of the research is to foster improved understanding of the cost weighting selection process and to provide practical, relevant design information to the flight control engineer.[16,17] If successful, the results may contribute to increased application and implementation of contemporary design techniques

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to advanced vehicle concepts requiring multiple, coordinated feedback loops.

Flight Dynamics Modeling

Within the commercial flight industry, there exists considerable interest for the development of a long-range, high-speed, high-capacity vehicle that can serve global transportation markets while remaining a viable concept both economically and environmentally.[18,19] The top end speeds of this vehicle, coupled with the use of composite materials for the primary structure, will lead to significant interaction between rigid-body and structural dynamics. Lowest frequency structural modes are expected to be well within one frequency decade (radian per second) of the short-period dynamics. This type of vehicle is a prime candidate for, and may necessarily require, a multivariable flight control system.[20] Therefore, an aeroelastic flight control application is chosen for the analytical expressions work presented here.

To capture the multidisciplinary aspects of rigid-body and vibrational motion, modeling efforts must return to the fundamental governing principles, such as in Ref. 21. The nonlinear dynamic model generated by this process can be linearized and reduced in order, resulting in a linear model appropriate for flight control design activities.

Reference 12 contains a model of this sort for a large, high-speed, elastic vehicle. Configuration geometry consists of a low-aspect-ratio swept wing, conventional aft tail, and small canard. A fifth-order polynomial matrix realization is given as

$$\left[\begin{array}{ccc|cc} s-(Z_\alpha/V_T) & -[1+(Z_q/V_T)]s & -(Z_{\dot\eta}/V_T)s-(Z_\eta/V_T) & 0 & 0\\ -M_\alpha & s^2-M_q s & -M_{\dot\eta}s-M_\eta & 0 & 0\\ -F_\alpha & -F_q s & s^2+(2\zeta\omega-F_{\dot\eta})s+\left(\omega^2-F_\eta\right) & 0 & 0\\ \hline 0 & -s & 0 & 1 & 0\\ 0 & -s & \phi' s & 0 & 1 \end{array}\right]\left[\begin{array}{c}\alpha(s)\\ \theta(s)\\ \eta(s)\\ \hline q(s)\\ q'(s)\end{array}\right]=\left[\begin{array}{cc} Z_{\delta_E}/V_T & Z_{\delta_C}/V_T\\ M_{\delta_E} & M_{\delta_C}\\ F_{\delta_E} & F_{\delta_C}\\ \hline 0 & 0\\ 0 & 0\end{array}\right]\left[\begin{array}{c}\delta_E(s)\\ \delta_C(s)\end{array}\right] \tag{1}$$

$$\boldsymbol{y}(s)=[q(s)\quad q'(s)]^T \qquad \boldsymbol{u}(s)=[\delta_E(s)\quad \delta_C(s)]^T \qquad \boldsymbol{y}(s)=G(s)\boldsymbol{u}(s) \qquad G(s)=\frac{1}{d(s)}\begin{bmatrix} n_{11}(s) & n_{12}(s)\\ n_{21}(s) & n_{22}(s)\end{bmatrix}$$

This model involves the small perturbation longitudinal dynamics of the closely spaced, effective short-period and first aeroelastic modes. Degrees of freedom include rigid-body angle of attack and pitch angle denoted as α and θ, whereas η corresponds to the generalized aeroelastic mode coordinate. Rigid pitch rate q and pitch rate sensed at the cockpit q' are the responses of interest. Control inputs consist of the elevator and canard deflections δ_E and δ_C. The reference flight condition is level flight at Mach 0.6 and altitude 5000 ft. Other parameters of interest in Eq. (1) are rigid and aeroelastic stability and control derivatives Z_i, M_i, F_i, with $i=\alpha, q, \eta, \dot\eta, \delta_E$, and δ_C; total flight velocity V_T; structural vibration frequency ω; damping ζ; and mode slope at the cockpit ϕ'. In Eq. (1), $G(s)$ denotes the vehicle transfer function matrix. Numerical values for the parameters are listed in Table 1.

Table 1 Vehicle numerical model definition

$Z_\alpha/V_T = -0.4158$ 1/s	$[1+(Z_q/V_T)] = 1.025$
$Z_\eta/V_T = -0.002666$ 1/s	$Z_{\dot\eta}/V_T = -0.0001106$ 1/s^2
$M_\alpha = -3.330$ 1/s^2	$M_q = -0.8302$ 1/s
$M_\eta = -0.06549$ 1/s^2	$M_{\dot\eta} = -0.003900$ 1/s
$F_\alpha = -1{,}040$ 1/s^2	$F_q = -78.35$ 1/s
$(2\zeta\omega - F_{\dot\eta}) = 0.6214$ 1/s	$(\omega^2 - F_\eta) = 34.83$ 1/s^2
$Z_{\delta_E}/V_T = -0.08021$ 1/s	$Z_{\delta_C}/V_T = -0.01644$ 1/s
$M_{\delta_E} = -5.115$ 1/s^2	$M_{\delta_C} = 0.8086$ 1/s^2
$F_{\delta_E} = -865.6$ 1/s^2	$F_{\delta_C} = -631.1$ 1/s^2
$\phi' = 0.02100$ ft/ft	
$n_{11} = -5.1s(s+0.33)(s-0.0031\pm j4.9)$	
$n_{12} = 0.81s(s+0.31)(s+1.9\pm j9.1)$	
$n_{21} = 13s(s+0.23)(s-3.4)(s+4.0)$	
$n_{22} = 14s(s+0.16)(s+0.66\pm j3.0)$	
$d = s(s+0.44\pm j1.2)(s+0.50\pm j6.0)$	
$\psi = -82s^2(s+0.35)$	

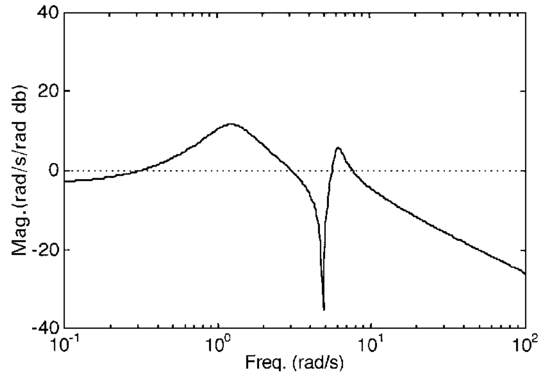

Fig. 1 Rigid pitch rate frequency response.

Table 1 lists the vehicle transfer function factors, whereas Figs. 1 and 2 show the q/δ_E and q'/δ_E frequency responses, respectively. For these same channels, Figs. 3 and 4 indicate the time responses due to a nose up 0.01-rad step command. As observed from the information given in Table 1 and Figs. 1–4, important open-loop dynamic deficiencies include the level of damping of the short-period and aeroelastic modes, as well as aeroelastic contributions to the dynamic responses. High levels of mode interaction are present. Because of the extreme levels of flexibility in the vehicle structure, the cockpit pitch rate response exhibits high-frequency transient motions and response reversal (nonminimum phaseness). The inherent airframe dynamics are unacceptable for manual control by the pilot, or for passenger ride comfort, and flight control augmentation is necessary.

Flight Control Design

In short, objectives of the flight control system are to correct the characteristics just noted, as well as provide satisfactory levels of robustness to modeling errors with feedback architecture as simple as possible. As an initial step into the analytical expressions work, consider the standard LQ state feedback strategy for flight control system development. Although better formulations for flight control design exist, the standard LQ approach is the most well-known contemporary design technique, and it provides a simple framework for illustrating the analytical expressions work.

Suppose the state-space description for the vehicle model in Eq. (1) is

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \qquad \boldsymbol{y}(t) = C\boldsymbol{x}(t) \tag{2}$$

Recall with A and B stabilizable and A and C detectable, the optimal state feedback control law, which minimizes a cost function blending the responses and inputs, or[2]

$$J = \int_0^\infty \{\boldsymbol{y}^T(\tau)Q\boldsymbol{y}(\tau) + \boldsymbol{u}^T(\tau)R\boldsymbol{u}(\tau)\}\,\mathrm{d}\tau \tag{3}$$

with symmetric, positive semidefinite Q and symmetric, positive definite R, is given as

$$\boldsymbol{u}(t) = \boldsymbol{u}_c(t) - K_R\boldsymbol{x}(t) \qquad K_R = R^{-1}B^TP \tag{4}$$

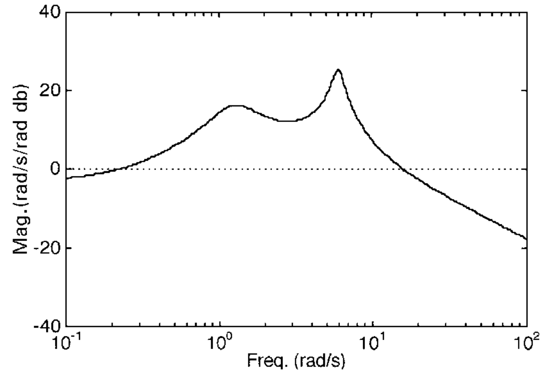

Fig. 2 Cockpit pitch rate frequency response.

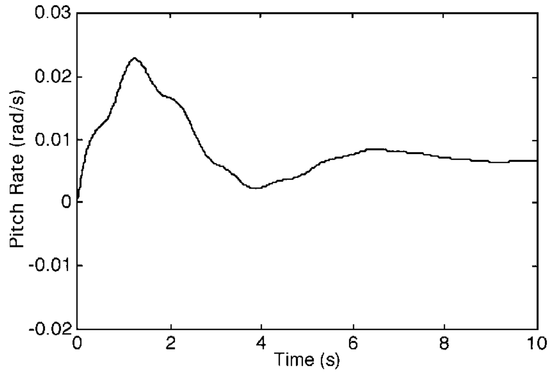

Fig. 3 Rigid pitch rate time response.

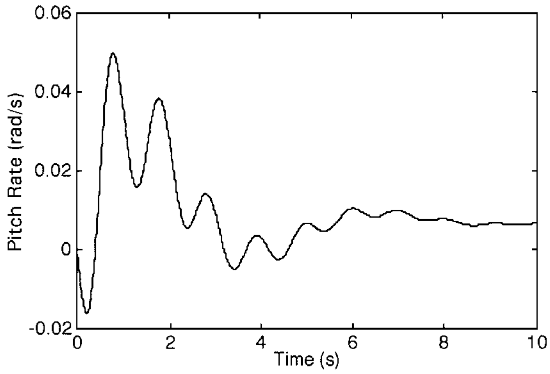

Fig. 4 Cockpit pitch rate time response.

The matrix P in Eq. (4) is the solution to the algebraic Riccati equation

$$PA + A^T P - PBR^{-1}B^T P + C^T QC = 0 \tag{5}$$

Finally, after considerable manipulation, it can be shown that the closed-loop poles are the values of s in the left-half plane, which satisfy

$$d(s)d(-s)\det[R + G^T(-s)QG(s)] = 0 \tag{6}$$

For the 2×2 aircraft system given in Eq. (1), the cost function weighting matrices are

$$Q = \mathrm{q}\begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \qquad R = \mathrm{r}\begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \tag{7}$$

where q and r represent overall weighting parameters.

Specification of the control input and response output set is a key step toward a successful flight control system. Selection of q' and δ_C in this set provides a close proximity actuator–sensor pair near the nose of the fuselage. This feedback channel should provide ample opportunity to suppress the harsh vibrational environment predicted at the cockpit. Also included in this set are q and δ_E. Historically, this pair of signals has proven to be effective in basic stability augmentation systems for the pitch axis.[4]

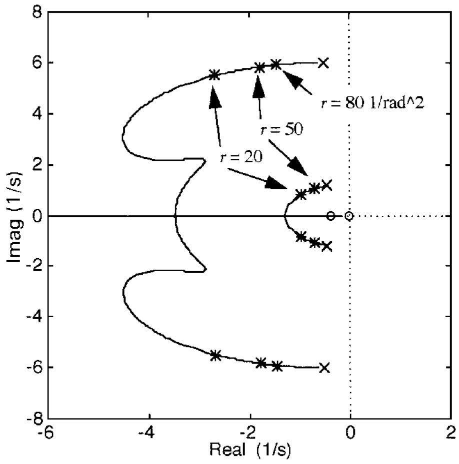

Fig. 5 Closed-loop pole migration dependency on overall control weight.

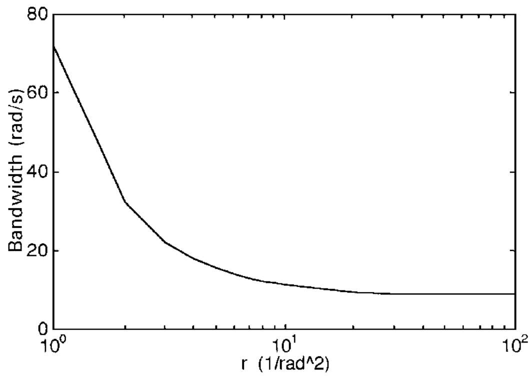

Fig. 6 Bandwidth dependency on overall control weight.

At this stage, the only freedom left unspecified is the weights Q and R. As seen from Figs. 2 and 4, the aeroelastic contamination at the cockpit station is severe. Therefore, the response cost $q_{22}q'^2$ will be scaled 2 to 1 with the $q_{11}q^2$ cost. Accordingly, to utilize the canard for the aeroelastic suppression role, the control cost $r_{22}\delta_C^2$ will be scaled 1 to 2 with the $r_{11}\delta_E^2$ cost. Diagonal Q and R will be enforced with $\mathrm{q} = 1$ and r used as an overall design knob to perform trades.

Figure 5 shows the closed-loop pole migration as the overall control weighting value for R is varied with

$$Q = 1\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \mathrm{s^2/rad^2} \qquad R = \mathrm{r}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \mathrm{1/rad^2} \tag{8}$$

Both short-period and aeroelastic mode damping can be augmented with decreased values of r. Figure 6 shows the resulting closed-loop bandwidth (q/δ_E magnitude crossover with dc value scaled to 0 dB) generated by a range of values for r. Recall high bandwidth is synonymous with low-stability robustness due to amplification of high-frequency modeling errors, such as neglected aeroelastic modes. High bandwidth also leads to increased actuator saturation with resulting time delay increments and increased probability for pilot induced oscillations. LQ state feedback guaranteed gain and phase margins are also in effect here, and if the design is carried to observer reconstruction of the state, loop transfer recovery concepts can be used to preserve these margins.

A value of $\mathrm{r} = 50$ provides an acceptable trade between damping augmentation and bandwidth limitations. The resulting closed-loop characteristic equation is

$$\{71s(s + 0.70 \pm j1.0)(s + 1.8 \pm j5.8)\}\{\cdots\} = 0 \tag{9}$$

and Figs. 7–10 are the closed-loop frequency and time responses corresponding to similar input conditions as in Figs. 1–4. In Eq. (9), $\{\cdots\}$ denotes the symmetric right-half plane roots, which are of no concern here. Significant improvements are seen, but it is noted

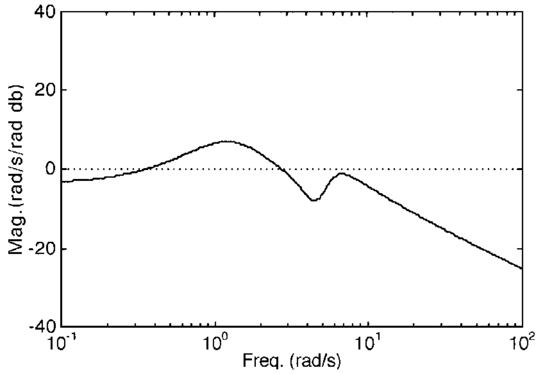

Fig. 7 Closed-loop rigid pitch rate frequency response.

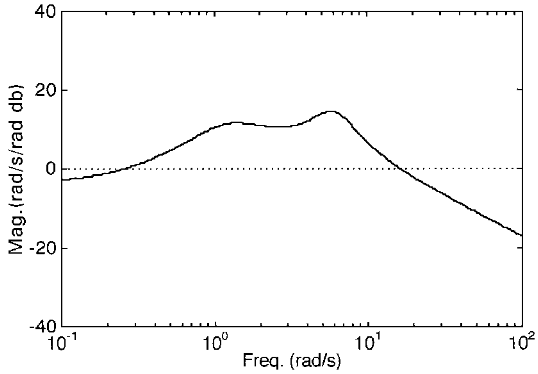

Fig. 8 Closed-loop cockpit pitch rate frequency response.

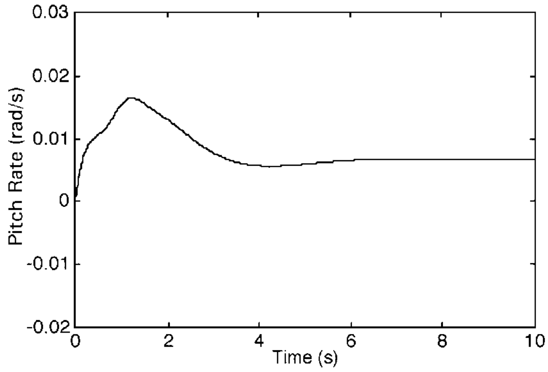

Fig. 9 Closed-loop rigid pitch rate time response.

the augmented characteristics may need further improvement for satisfactory handling qualities.

Other than some general knowledge about trends as the design knob $\mathfrak{r}$ is adjusted, the control system is largely numerical based and the flight control engineer is left with little understanding. Common questions facing the designer include how the individual weights affect particular modes, how the control weights and airframe parameters combine to yield key closed-loop dynamic features, how the properties change with flight condition, how the sensor location enters into the characteristics, etc. Further, inevitable design modifications will arise and assistance in implementing a tuning strategy that is short of a purely iterative approach is highly desirable. Answers to these issues could be obtained from a numerical-based sensitivity study. However, the conclusions would be problem specific, requiring additional calculations at other flight conditions, for example. Attention will now focus on analytical strategies for attacking these issues.

Closed-Loop Approximate Factoring

A technique is presented here to obtain the approximate functional dependence of the LQ characteristic factors on the basic aircraft parameters appearing in the governing differential equations and the cost function weights, for the purpose of obtaining insight into the closed-loop features. To accomplish this insight, a truncated Taylor

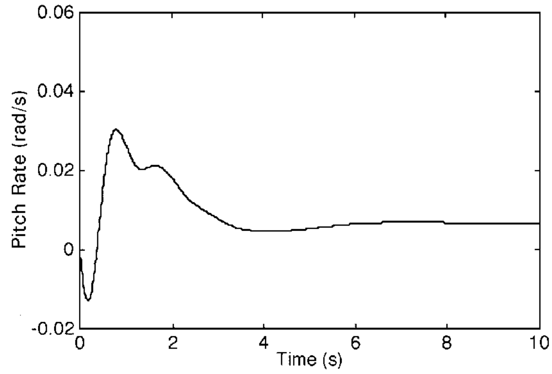

Fig. 10 Closed-loop cockpit pitch rate time response.

series is used as the tool to obtain the functional dependence. Relatively simple expressions appear to result with satisfactory accuracy.

Reconsider the LQ characteristic equation given in Eq. (6). Upon expansion of this determinant for the 2×2 aircraft system under consideration, Eq. (6) becomes

$$\begin{aligned}\phi &= \mathfrak{r}^2\left(r_{11}r_{22} - r_{12}^2\right)d\bar{d} + \mathfrak{r}\mathfrak{q}\{r_{11}q_{11}(n_{12}\bar{n}_{12}) + r_{11}q_{12}(n_{12}\bar{n}_{22} \\ &\quad + n_{22}\bar{n}_{12}) + r_{11}q_{22}(n_{22}\bar{n}_{22}) - r_{12}q_{11}(n_{11}\bar{n}_{12} + n_{12}\bar{n}_{11}) \\ &\quad - r_{12}q_{12}(n_{11}\bar{n}_{22} + n_{12}\bar{n}_{21} + n_{21}\bar{n}_{12} + n_{22}\bar{n}_{11}) - r_{12}q_{22}(n_{21}\bar{n}_{22} \\ &\quad + n_{22}\bar{n}_{21}) + r_{22}q_{11}(n_{11}\bar{n}_{11}) + r_{22}q_{12}(n_{11}\bar{n}_{21} + n_{21}\bar{n}_{11}) \\ &\quad + r_{22}q_{22}(n_{21}\bar{n}_{21})\} + \mathfrak{q}^2\left(q_{11}q_{22} - q_{12}^2\right)\psi\bar{\psi} = 0 \end{aligned} \tag{10}$$

Equation (10) has made use of the transmission zero polynomial ψ, or

$$\frac{n_{11}n_{22} - n_{12}n_{21}}{d} = \psi \tag{11}$$

In Eq. (10), $\phi(s)$ denotes the LQ characteristic polynomial and the overbar denotes evaluation at $-s$.

Note at this stage that the aircraft numerator and denominator polynomials appearing in Eq. (10) are easily generated by applying Cramer's rule to Eq. (1). These polynomials can be expressed as

$$n_{ij}(s) = \sum_{k=1}^{4!} \mathfrak{n}_k(s) \qquad d(s) = \sum_{k=1}^{3!} \mathfrak{d}_k(s) \tag{12}$$

The $\mathfrak{n}_k(s)$ and $\mathfrak{d}_k(s)$ terms in Eq. (12) consist of products of second order, first order, and constant polynomials, which appear as matrix elements in Eq. (1) having the basic aircraft parameters as coefficients. As will be shown, the factored nature of the $\mathfrak{n}_k(s)$ and $\mathfrak{d}_k(s)$ terms is key to obtaining preliminary approximations to the factors of $\phi(s)$ (Ref. 12).

Using Eq. (12), $\phi(s)$ can be expressed in expanded form as

$$\phi(s) = \phi_{10}s^{10} + \phi_8 s^8 + \phi_6 s^6 + \phi_4 s^4 + \phi_2 s^2 \tag{13}$$

Further, $\phi(s)$ can be expressed in factored form as

$$\begin{aligned}\phi(s) &= \left\{ks\left[s^2 + (2\zeta\omega)_1 s + \omega_1^2\right]\left[s^2 + (2\zeta\omega)_2 s + \omega_2^2\right]\right\} \\ &\quad \times \left\{-ks\left[s^2 - (2\zeta\omega)_1 s + \omega_1^2\right]\left[s^2 - (2\zeta\omega)_2 s + \omega_2^2\right]\right\}\end{aligned} \tag{14}$$

where ω_i^2 and $(2\zeta\omega)_i$ are the LQ characteristic polynomial factors and k is the polynomial gain. Here, to match the closed-loop design given in Eq. (9), the factors are stipulated as two complex conjugate and one at the origin. This assumption does not restrict the method from other applications involving real or mixed real-complex conjugate roots. In general, to know how many zero, nonzero real, and complex-conjugate factors $\phi(s)$ has, as in Eq. (14), the associated numerical model must already be factored.

To obtain preliminary approximations to the factors in Eq. (14), select one or more of the components [see Eq. (12)] of the $n_{ij}\bar{n}_{pq}$ and $d\bar{d}$ terms from Eq. (10) that best satisfy the following two criteria.

1) The analytical expressions for the preliminary factors should have identical root constituency as that in Eq. (14).

2) The numerical values for the preliminary factors should be as close as possible to the numerical values for the exact factors in Eq. (14). Denote these preliminary factors as

$$\tilde{\phi}(s) = \left\{\tilde{k}s\left[s^2 + (\widetilde{2\zeta\omega})_1 s + \tilde{\omega}_1^2\right]\left[s^2 + (\widetilde{2\zeta\omega})_2 s + \tilde{\omega}_2^2\right]\right\}$$
$$\times \left\{-\tilde{k}s\left[s^2 - (\widetilde{2\zeta\omega})_1 s + \tilde{\omega}_1^2\right]\left[s^2 - (\widetilde{2\zeta\omega})_2 s + \tilde{\omega}_2^2\right]\right\}$$
$$= \tilde{\phi}_{10}s^{10} + \tilde{\phi}_8 s^8 + \tilde{\phi}_6 s^6 + \tilde{\phi}_4 s^4 + \tilde{\phi}_2 s^2 \tag{15}$$

Now focus on the technique that generates analytical corrections to the preliminary factors, thereby increasing the accuracy of the approximate symbolic expressions for the factors. The functional dependence of the polynomial coefficients on the polynomial factors is readily available by expanding Eq. (14) and equating like powers of s with Eq. (13) or

$$\phi_i = f_i\left[k, \omega_1^2, (2\zeta\omega)_1, \omega_2^2, (2\zeta\omega)_2\right] \tag{16}$$

where f_i is a continuous, nonlinear function consisting of addition and multiplication of the factors. Of more importance here is the development of the functional dependence of the factors on the coefficients, obtained by inverting Eq. (16).

Consider expressing the continuous, functional dependence of the coefficients on the factors, as given in Eq. (16), in a Taylor series,

$$\phi_i = \phi_i|_0 + \left.\frac{\partial\phi_i}{\partial \boldsymbol{x}}\right|_0 (\boldsymbol{x} - \boldsymbol{x}|_0) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}|_0)^T$$
$$\times \left.\frac{\partial^2\phi_i}{\partial x^2}\right|_0 (\boldsymbol{x} - \boldsymbol{x}|_0) + \cdots$$
$$= \phi_i|_0 + \left.\frac{\partial\phi_i}{\partial \boldsymbol{x}}\right|_0 \Delta\boldsymbol{x} + \frac{1}{2}\Delta\boldsymbol{x}^T \left.\frac{\partial^2\phi_i}{\partial x^2}\right|_0 \Delta\boldsymbol{x} + \cdots \tag{17}$$

where the independent variables are the factors contained in the column vectors $\boldsymbol{x}$ given as

$$\boldsymbol{x} = \left[k \quad \omega_1^2 \quad (2\zeta\omega)_1 \quad \omega_2^2 \quad (2\zeta\omega)_2\right]^T \tag{18}$$

the first partial derivative $\partial\phi_i/\partial\boldsymbol{x}$ is a row vector defined as

$$\frac{\partial\phi_i}{\partial\boldsymbol{x}} = \left[\frac{\partial f_i}{\partial k} \quad \frac{\partial f_i}{\partial\omega_1^2} \quad \frac{\partial f_i}{\partial(2\zeta\omega)_1} \quad \frac{\partial f_i}{\partial\omega_2^2} \quad \frac{\partial f_i}{\partial(2\zeta\omega)_2}\right] \tag{19}$$

the second partial derivative $\partial^2\phi_i/\partial x^2$ is a matrix defined as

$$\frac{\partial^2\phi_i}{\partial x^2} = \begin{bmatrix} \dfrac{\partial^2 f_i}{\partial k^2} & \cdots & \dfrac{\partial^2 f_i}{\partial k\,\partial(2\zeta\omega)_2} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f_i}{\partial(2\zeta\omega)_2\,\partial k} & \cdots & \dfrac{\partial^2 f_i}{\partial(2\zeta\omega)_2^2} \end{bmatrix} \tag{20}$$

and the notation $|_0$ denotes the nominal values about which the Taylor series is expanded. Also, $\Delta\boldsymbol{x}$ is defined as

$$\Delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}|_0 = \left[\Delta k \quad \Delta\omega_1^2 \quad \Delta(2\zeta\omega)_1 \quad \Delta\omega_2^2 \quad \Delta(2\zeta\omega)_2\right]^T \tag{21}$$

For a given numerical model, the basic system parameters appearing as matrix elements in Eqs. (1) and (7), and the corresponding polynomial coefficients, have specific numerical values associated with them. The functional dependence of the coefficients on the factors can be expressed in a Taylor series expanded about these specific numerical values, but a Taylor series, in particular Eq. (17), is more general than this. The Taylor series in Eq. (17) can be expanded about any set of numerical values for the basic system parameters.

In Eq. (17), the nominal condition $|_0$ is specified to be a set of numerical values for the basic parameters that result in $\phi_i|_0 = \tilde{\phi}_i$, i.e., after the full expressions for the polynomial coefficients are evaluated, the coefficients corresponding to the preliminary factors in Eq. (15) result. In other words, the system parameters appearing in the specific $\mathfrak{n}_k(s)$ and $\mathfrak{d}_k(s)$ terms that form the preliminary factors, as well as the cost function weights appearing as overall coefficients, are evaluated at their given values, whereas the remaining system parameters are evaluated at zero, which, in general, is not necessarily their given value. When the system parameters are evaluated in this way, the $\mathfrak{n}_k(s)$ and $\mathfrak{d}_k(s)$ terms excluded from the preliminary factors must equal zero. This places a third criterion on the selection of the preliminary factors.

3) The remaining $\mathfrak{n}_k(s)$ and $\mathfrak{d}_k(s)$ terms, and associated cost function weights, excluded from the preliminary factors must be a function of at least one system parameter that does not appear in the preliminary factor terms.

If all of the coefficient Taylor series in Eq. (17) are combined and rearranged, a single vector Taylor series results:

$$\Delta\boldsymbol{\phi} = A\Delta\boldsymbol{x} + \cdots \tag{22}$$

where $\Delta\boldsymbol{\phi}$ and A are given as

$$\Delta\boldsymbol{\phi} = [(\phi_2 - \phi_2|_0)(\phi_4 - \phi_4|_0)(\phi_6 - \phi_6|_0)$$
$$(\phi_8 - \phi_8|_0)(\phi_{10} - \phi_{10}|_0)]^T \tag{23}$$
$$A = \left[\left.\frac{\partial\phi_2}{\partial\boldsymbol{x}}\right|_0^T \quad \left.\frac{\partial\phi_4}{\partial\boldsymbol{x}}\right|_0^T \quad \left.\frac{\partial\phi_6}{\partial\boldsymbol{x}}\right|_0^T \quad \left.\frac{\partial\phi_8}{\partial\boldsymbol{x}}\right|_0^T \quad \left.\frac{\partial\phi_{10}}{\partial\boldsymbol{x}}\right|_0^T\right]^T$$

Note that A is a square matrix.

The matrix A can be inverted analytically to obtain corrections to the preliminary factors:

$$\Delta\boldsymbol{x} \approx A^{-1}\Delta\boldsymbol{\phi} \tag{24}$$

where the higher-order terms in the series have been neglected. The matrix A will be invertible as long as its determinant, when evaluated numerically, is nonzero. This inversion calculation requires the symbolic expressions for $\Delta\phi$ obtained from the difference between Eqs. (13) and (15), as well as the symbolic expressions for A obtained from differentiation of Eq. (16). The approximate analytical expressions for the LQ characteristic polynomial factors are finally obtained by summing the preliminary factors in Eq. (15) with their corrections in Eq. (24):

$$\boldsymbol{x} \approx \boldsymbol{x}|_0 + \Delta\boldsymbol{x} \tag{25}$$

At this stage, the symbolic expressions for the factors, as functions of the basic system parameters, may be quite cumbersome for obtaining insight into key dynamic characteristics. By neglecting small terms based on their relative numerical magnitude, the symbolic expressions can often be simplified to a level that is conducive to understanding major players in the dynamic characteristics.

Factoring Example

To demonstrate the approximate factoring technique just presented and to obtain insight into the LQ flight control system design given earlier, reconsider the vehicle model presented in Eq. (1). By applying Cramer's rule to Eq. (1), the vehicle polynomials $d(s)$ and $n_{ij}(s)$, with the summation structure of Eq. (12), are generated. As an example, by expanding about the third row, $d(s)$ is

$$d(s) = -F_\alpha[1 + (Z_q/V_T)]s(M_{\dot\eta}s + M_\eta) - F_\alpha\left(s^2 - M_q s\right)$$
$$\times [(Z_{\dot\eta}/V_T)s + (Z_\eta/V_T)] - F_q s[s - (Z_\alpha/V_T)]$$
$$\times (M_{\dot\eta}s + M_\eta) - F_q s M_\alpha[(Z_{\dot\eta}/V_T)s + (Z_\eta/V_T)]$$
$$+ \left\{s^2 + (2\zeta\omega - F_{\dot\eta})s + \left(\omega^2 - F_\eta\right)\right\}[s - (Z_\alpha/V_T)]$$
$$\times \left(s^2 - M_q s\right) - \left\{s^2 + (2\zeta\omega - F_{\dot\eta})s + \left(\omega^2 - F_\eta\right)\right\}$$
$$\times [1 + (Z_q/V_T)]sM_\alpha \tag{26}$$

Other polynomials are not shown for conciseness. These polynomials combine according to Eq. (10) to form the LQ characteristic polynomial. Further, the exact numerical solution for this closed-loop polynomial is

$$\phi(s) = \{71s(s^2 + 1.4s + 1.6)(s^2 + 3.5s + 37)\}\{\cdots\} \tag{27}$$

Out of all possible combinations of the factored terms in Eq. (10) that factor symbolically into one real root at the origin and two complex conjugate roots (criterion 1) and are as close to the exact numerical values as possible (criterion 2), the following terms are selected as the preliminary factors:

$$\tilde{\phi}(s) = \Big(\Upsilon(r_{11}r_{22})^{\frac{1}{2}} s\big\{s^2 + [-(Z_\alpha/V_T) - M_q]s + [(Z_\alpha/V_T)M_q - [1 + (Z_q/V_T)]M_\alpha]\big\}\big[s^2 + (2\zeta\omega - F_{\dot\eta})s + \big(\omega^2 - F_\eta\big)\big]\Big)\{\cdots\}$$
$$= \{71s(s^2 + 1.2s + 3.8)(s^2 + 0.62s + 35)\}\{\cdots\} \qquad (28)$$

Note the factors in Eq. (28) originate from the last two of the six components within the bare airframe denominator polynomial $d(s)$ in Eq. (26). Comparisons of Eqs. (27) and (28) indicate the preliminary factors are not, by themselves, sufficiently accurate, implying corrections are needed. Specifically, ω_1^2 (1.6 vs 3.8) and $(2\zeta\omega)_2$ (3.5 vs 0.62) indicate large discrepancies. With review of the basic parameters, it can be verified that criterion 3 is met for the selection.

Before moving on to the correction terms, a slight modification to the strategy is considered. In Eq. (10), the Υ^2 term denotes the open-loop root locations, whereas the Υq and q^2 terms represent the feedback effects that modify the open-loop roots into closed-loop roots. Note the chosen preliminary factors in Eq. (28) are the classic short-period and aerodynamically damped pure vibration factors, arising solely from the Υ^2 term. Numerically speaking, the preliminary factors are not a sufficient approximation to even the bare airframe poles in Table 1:

$$d(s) = s(s^2 + 0.88s + 1.6)(s^2 + 1.0s + 36) \qquad (29)$$

As proposed, the factoring technique would attempt to correct for errors originating from both open-loop and closed-loop sources in one step. Better accuracy can be had if the problem is broken down into a two-step procedure, first correcting for the open-loop errors and second for the feedback errors. The first step was the subject of Ref. 12, and the results are applicable here. By applying the method to the airframe only, the expressions for the new preliminary factors become

$$\tilde{k} = \Upsilon(r_{11}r_{22})^{\frac{1}{2}} = 71$$
$$\tilde{\omega}_1^2 = (Z_\alpha/V_T)M_q - [1 + (Z_q/V_T)]M_\alpha - \frac{[1 + (Z_q/V_T)]M_\eta F_\alpha}{\big(\omega^2 - F_\eta\big)} = 1.8$$
$$(\widetilde{2\zeta\omega})_1 = -(Z_\alpha/V_T) - M_q - \frac{(\{(Z_\eta/V_T) + [1 + (Z_q/V_T)]M_{\dot\eta}\}F_\alpha + M_\eta F_q)}{\big(\omega^2 - F_\eta\big)} = 0.90$$
$$\tilde{\omega}_2^2 = \big(\omega^2 - F_\eta\big) + \frac{[1 + (Z_q/V_T)]M_\eta F_\alpha}{\big(\omega^2 - F_\eta\big)} = 37$$
$$(\widetilde{2\zeta\omega})_2 = (2\zeta\omega - F_{\dot\eta}) + \frac{(\{(Z_\eta/V_T) + [1 + (Z_q/V_T)]M_{\dot\eta}\}F_\alpha + M_\eta F_q)}{\big(\omega^2 - F_\eta\big)} = 0.97 \qquad (30)$$

These modified preliminary factors provide a much better approximation to the airframe factors in Eq. (29). To account for the effect of feedback augmentation on the root locations, corrections to these factors will now be considered, where only the Υq and q^2 terms in Eq. (10) are utilized.

To construct the symbolic Taylor series for the polynomial coefficients, consider the functional dependence of the polynomial coefficients on the polynomial factors from Eqs. (13) and (14),

$$\phi_2 = -k^2\big\{\omega_1^4\omega_2^4\big\}$$
$$\phi_4 = -k^2\big\{2\omega_1^4\omega_2^2 + 2\omega_2^4\omega_1^2 - \omega_1^4(2\zeta\omega)_2^2 - \omega_2^4(2\zeta\omega)_1^2\big\}$$
$$\phi_6 = -k^2\big\{\omega_1^4 + \omega_2^4 + 4\omega_1^2\omega_2^2 - 2\omega_1^2(2\zeta\omega)_2^2 - 2\omega_2^2(2\zeta\omega)_1^2 + (2\zeta\omega)_1^2(2\zeta\omega)_2^2\big\} \qquad (31)$$
$$\phi_8 = -k^2\big\{2\omega_1^2 + 2\omega_2^2 - (2\zeta\omega)_1^2 - (2\zeta\omega)_2^2\big\} \qquad \phi_{10} = -k^2$$

From Eq. (31) an exact expression for k in closed form is available. Thus, no correction is required for k and the corresponding row for ϕ_{10} in Eq. (22) is dropped and A becomes 4×4.

After calculating the appropriate partial derivatives of ϕ_i, the matrix A is expressed as

$$A = -2\tilde{k}^2\begin{bmatrix} \tilde{\omega}_2^4\tilde{\omega}_1^2 & 0 & \tilde{\omega}_1^4\tilde{\omega}_2^2 & 0 \\ \tilde{\omega}_2^4 + \big\{2\tilde{\omega}_2^2 - (\widetilde{2\zeta\omega})_2^2\big\}\tilde{\omega}_1^2 & -\tilde{\omega}_2^4(\widetilde{2\zeta\omega})_1 & \tilde{\omega}_1^4 + \big\{2\tilde{\omega}_1^2 - (\widetilde{2\zeta\omega})_1^2\big\}\tilde{\omega}_2^2 & -\tilde{\omega}_1^4(\widetilde{2\zeta\omega})_2 \\ 2\tilde{\omega}_2^2 + \tilde{\omega}_1^2 - (\widetilde{2\zeta\omega})_2^2 & -\big\{2\tilde{\omega}_2^2 - (\widetilde{2\zeta\omega})_2^2\big\}(\widetilde{2\zeta\omega})_1 & 2\tilde{\omega}_1^2 + \tilde{\omega}_2^2 - (\widetilde{2\zeta\omega})_1^2 & -\big\{2\tilde{\omega}_1^2 - (\widetilde{2\zeta\omega})_1^2\big\}(\widetilde{2\zeta\omega})_2 \\ 1 & -(\widetilde{2\zeta\omega})_1 & 1 & -(\widetilde{2\zeta\omega})_2 \end{bmatrix} \qquad (32)$$

After some effort, the A matrix can be inverted analytically. With the goal of simple and useful expressions, A^{-1} has far too many terms to retain. For example, the 1,1 element of A^{-1} is given as

$$A_{11}^{-1} = \frac{8\big\{-\tilde{\omega}_2^6(\widetilde{2\zeta\omega})_1(\widetilde{2\zeta\omega})_2 + \big[-3\tilde{\omega}_1^4 - (\widetilde{2\zeta\omega})_1^4 + 4\tilde{\omega}_1^2(\widetilde{2\zeta\omega})_1^2\big]\tilde{\omega}_2^2(\widetilde{2\zeta\omega})_1(\widetilde{2\zeta\omega})_2 - \big[-2\tilde{\omega}_1^2 + (\widetilde{2\zeta\omega})_1^2\big]\big[2\tilde{\omega}_2^2(\widetilde{2\zeta\omega})_1 - (\widetilde{2\zeta\omega})_2^2(\widetilde{2\zeta\omega})_1\big]\tilde{\omega}_2^2(\widetilde{2\zeta\omega})_2\big\}}{16\tilde{k}^2\tilde{\omega}_1^2\tilde{\omega}_2^2(\widetilde{2\zeta\omega})_1(\widetilde{2\zeta\omega})_2\big\{\tilde{\omega}_1^8 + \tilde{\omega}_2^8 + 6\tilde{\omega}_1^4\tilde{\omega}_2^4 - \big[\tilde{\omega}_2^4 + \tilde{\omega}_1^4\big]4\tilde{\omega}_1^2\tilde{\omega}_2^2 + \tilde{\omega}_2^4(\widetilde{2\zeta\omega})_1^4 + \tilde{\omega}_1^4(\widetilde{2\zeta\omega})_2^4 - \big[\tilde{\omega}_2^2(\widetilde{2\zeta\omega})_1^2 + \tilde{\omega}_1^2(\widetilde{2\zeta\omega})_2^2\big]4\tilde{\omega}_1^2\tilde{\omega}_2^2 + \big[2\tilde{\omega}_2^2 - (\widetilde{2\zeta\omega})_2^2\big]\tilde{\omega}_1^4(\widetilde{2\zeta\omega})_1^2 + \big[2\tilde{\omega}_1^2 - (\widetilde{2\zeta\omega})_1^2\big]\tilde{\omega}_2^4(\widetilde{2\zeta\omega})_2^2 + 2\tilde{\omega}_2^6(\widetilde{2\zeta\omega})_1^2 + 2\tilde{\omega}_1^6(\widetilde{2\zeta\omega})_2^2\big\}} \qquad (33)$$

However, if small numerical terms are neglected, the 1,1 element of A^{-1}, as well as all other elements, can be reasonably approximated by

$$A^{-1} \approx \frac{1}{\Gamma}\begin{bmatrix} -\big\{4\tilde{\omega}_1^2 - \tilde{\omega}_2^2\big\}\tilde{\omega}_2^2(\widetilde{2\zeta\omega})_1(\widetilde{2\zeta\omega})_2 & 2\tilde{\omega}_1^4\tilde{\omega}_2^2(\widetilde{2\zeta\omega})_1(\widetilde{2\zeta\omega})_2 & -\tilde{\omega}_1^4\tilde{\omega}_2^4(\widetilde{2\zeta\omega})_1(\widetilde{2\zeta\omega})_2 & 2\tilde{\omega}_1^6\tilde{\omega}_2^4(\widetilde{2\zeta\omega})_1(\widetilde{2\zeta\omega})_2 \\ -\big\{4\tilde{\omega}_1^2 - \tilde{\omega}_2^2\big\}\tilde{\omega}_2^2(\widetilde{2\zeta\omega})_2 & -2\tilde{\omega}_2^4\tilde{\omega}_1^2(\widetilde{2\zeta\omega})_2 & \tilde{\omega}_1^4\tilde{\omega}_2^4(\widetilde{2\zeta\omega})_2 & -\big\{\tilde{\omega}_1^2 - 3(\widetilde{2\zeta\omega})_1^2\big\}\tilde{\omega}_1^4\tilde{\omega}_2^4(\widetilde{2\zeta\omega})_2 \\ 2\tilde{\omega}_1^2\tilde{\omega}_2^2(\widetilde{2\zeta\omega})_1(\widetilde{2\zeta\omega})_2 & -2\tilde{\omega}_2^4\tilde{\omega}_1^2(\widetilde{2\zeta\omega})_1(\widetilde{2\zeta\omega})_2 & \tilde{\omega}_2^6\tilde{\omega}_1^2(\widetilde{2\zeta\omega})_1(\widetilde{2\zeta\omega})_2 & -2\tilde{\omega}_2^6\tilde{\omega}_1^4(\widetilde{2\zeta\omega})_1(\widetilde{2\zeta\omega})_2 \\ -\big\{2\tilde{\omega}_1^2 - \tilde{\omega}_2^2\big\}\tilde{\omega}_1^2(\widetilde{2\zeta\omega})_1 & -2\tilde{\omega}_2^4\tilde{\omega}_1^2(\widetilde{2\zeta\omega})_1 & \tilde{\omega}_2^6\tilde{\omega}_1^2(\widetilde{2\zeta\omega})_1 & \big\{4\tilde{\omega}_1^2 - \tilde{\omega}_2^2\big\}\tilde{\omega}_2^6\tilde{\omega}_1^2(\widetilde{2\zeta\omega})_1 \end{bmatrix}$$

$$= \begin{bmatrix} -1.2e{-8} & -1.8e{-8} & 3.1e{-7} & -2.3e{-6} \\ -1.0e{-8} & 6.6e{-8} & -2.5e{-7} & -2.2e{-7} \\ -4.7e{-9} & 1.6e{-7} & -2.9e{-6} & 2.2e{-5} \\ -3.0e{-9} & 1.3e{-7} & -4.6e{-6} & 9.1e{-5} \end{bmatrix} \qquad \Gamma = -2\tilde{k}^2\big\{\tilde{\omega}_2^8\tilde{\omega}_1^2(\widetilde{2\zeta\omega})_1(\widetilde{2\zeta\omega})_2\big\} \qquad (34)$$

Moving on to the calculation for the excluded terms not selected in the preliminary factors, $\Delta\phi$ consists of the rq and q^2 terms in Eq. (10), broken down into coefficients for the relevant powers of s. Recall that terms originating from the r^2 set are not included in $\Delta\phi$ for the modified two-step strategy. When fully expanded, expressions for $\Delta\phi_i$ are extremely long and unsatisfactory for obtaining insight. After retaining only the numerically dominant terms similar to the transition step from Eq. (33) to Eq. (34), $\Delta\phi$ is approximately

$$\begin{aligned}
\Delta\phi_2 &\approx \mathsf{rq}\Big\{-r_{11}q_{22}\big[-(Z_\alpha/V_T)M_\eta F_{\delta_C}\big]^2 \\
&\quad - r_{22}(q_{11}+q_{22})\big[-(Z_\alpha/V_T)M_\eta F_{\delta_E} \\
&\quad - (Z_\alpha/V_T)M_{\delta_E}\big[\omega^2 - F_\eta\big]\big]^2\Big\} = -4.4e{+}5 \\
\Delta\phi_4 &\approx \mathsf{rq}\Big\{r_{11}q_{22}\Big(M_\eta F_{\delta_C} + M_{\delta_C}\big[\omega^2 - F_\eta\big] \\
&\quad + [1 + (Z_q/V_T)]M_\alpha F_{\delta_C}\phi'\Big)^2 \\
&\quad + r_{22}q_{22}\Big(M_{\delta_E}\big[\omega^2 - F_\eta\big]\Big)^2\Big\} = 5.8e{+}6 \\
\Delta\phi_6 &\approx \mathsf{rq}\Big\{-r_{11}q_{22}2F_{\delta_C}\phi'\Big(M_\eta F_{\delta_C} + M_{\delta_C}\big[\omega^2 - F_\eta\big] \\
&\quad + [1 + (Z_q/V_T)]M_\alpha F_{\delta_C}\phi'\Big) \\
&\quad + r_{22}q_{22}2\Big(M_{\delta_E} - F_{\delta_E}\phi'\Big)M_{\delta_E}\big[\omega^2 - F_\eta\big]\Big\} = 1.4e{+}5 \\
\Delta\phi_8 &\approx \mathsf{rq}\Big\{r_{11}q_{22}\Big(-F_{\delta_C}\phi'\Big)^2 + r_{22}q_{22}\Big(M_{\delta_E} - F_{\delta_E}\phi'\Big)^2\Big\} \\
&= 5.2e{+}4
\end{aligned} \tag{35}$$

Corrections to the preliminary factors Δx are obtained by the product of A^{-1} and $\Delta\phi$. Even with the already simplified results for A^{-1} and $\Delta\phi$ in Eqs. (34) and (35), the Δx expressions can be extremely long when fully expanded. Again, retaining only the numerically dominant terms, the corrections for the preliminary factors are

$$\begin{aligned}
\Delta\omega_1^2 &\approx \frac{\tilde{\omega}_1^2}{-\tilde{k}^2\tilde{\omega}_2^6}\Big(\Delta\phi_4 + \tilde{\omega}_1^2\tilde{\omega}_2^2\Delta\phi_8\Big) = -0.22 \\
\Delta(2\zeta\omega)_1 &\approx \frac{1}{2\tilde{k}^2\tilde{\omega}_2^4\widetilde{(2\zeta\omega)}_1}\Delta\phi_4 = 0.38 \\
\Delta\omega_2^2 &\approx \frac{1}{\tilde{k}^2\tilde{\omega}_2^4}\Big(\Delta\phi_4 + \tilde{\omega}_1^2\tilde{\omega}_2^2\Delta\phi_8\Big) = 2.1 \\
\Delta(2\zeta\omega)_2 &\approx \frac{4\tilde{\omega}_1^2 - \tilde{\omega}_2^2}{-2\tilde{k}^2\tilde{\omega}_2^2\widetilde{(2\zeta\omega)}_2}\Delta\phi_8 = 4.8
\end{aligned} \tag{36}$$

The pinnacle result from the preceding steps is now available. Approximate analytical expressions for the LQ characteristic polynomial factors are

$$\begin{aligned}
\omega_1^2 &\approx \tilde{\omega}_1^2 + \Delta\omega_1^2 = 1.5 \\
(2\zeta\omega)_1 &\approx \widetilde{(2\zeta\omega)}_1 + \Delta(2\zeta\omega)_1 = 1.3 \\
\omega_2^2 &\approx \tilde{\omega}_2^2 + \Delta\omega_2^2 = 39 \\
(2\zeta\omega)_2 &\approx \widetilde{(2\zeta\omega)}_2 + \Delta(2\zeta\omega)_2 = 5.7
\end{aligned} \tag{37}$$

where, ultimately, every term has been traced back to a function of the basic vehicle parameters and cost function weightings using Eqs. (30), (35), and (36). Table 2 contains the expanded results.

Before ever using the approximate expressions, first consider their accuracy. If all terms in the preceding calculations were retained, the approximate expressions would be intractable for practical applications. In most physical systems, as is the case here, all terms are not significant. Small terms, based on relative numerical magnitude, were neglected. In other words, accuracy is being traded off for simplicity. Here the trade is reasonably acceptable as seen by contrasting the values in Eq. (37) with Eq. (27). The approximate expressions are roughly accurate allowing underlying relationships to be explored.

Table 2 Analytical expressions for LQ eigenvalues

$$\begin{aligned}
\omega_1^2 &\approx \frac{Z_\alpha}{V_T}M_q - \left(1 + \frac{Z_q}{V_T}\right)M_\alpha - \frac{[1 + (Z_q/V_T)]M_\eta F_\alpha}{\big(\omega^2 - F_\eta\big)} \\
&\quad + \frac{\tilde{\omega}_1^2}{-\tilde{k}^2\tilde{\omega}_2^6}\Big[\mathsf{rq}\Big\{r_{11}q_{22}\Big(M_\eta F_{\delta C} + M_{\delta C}\big[\omega^2 - F_\eta\big] \\
&\quad + [1 + (Z_q/V_T)]M_\alpha F_{\delta C}\phi'\Big)^2 + r_{22}q_{22}\Big(M_{\delta_E}\big[\omega^2 - F_\eta\big]\Big)^2\Big\} \\
&\quad + \tilde{\omega}_1^2\tilde{\omega}_2^2\mathsf{rq}\Big\{r_{11}q_{22}\Big(-F_{\delta C}\phi'\Big)^2 + r_{22}q_{22}\Big(M_{\delta_E} - F_{\delta_E}\phi'\Big)^2\Big\}\Big] \\
(2\zeta\omega)_1 &\approx -\frac{Z_\alpha}{V_T} - M_q - \frac{\{(Z_\eta/V_T) + [1 + (Z_q/V_T)]M_{\dot{\eta}}\}F_\alpha + M_\eta F_q}{\big(\omega^2 - F_\eta\big)} \\
&\quad + \frac{1}{2\tilde{k}^2\tilde{\omega}_2^4\widetilde{(2\zeta\omega)}_1}\mathsf{rq}\Big\{r_{11}q_{22}\Big(M_\eta F_{\delta C} + M_{\delta C}\big[\omega^2 - F_\eta\big] \\
&\quad + [1 + (Z_q/V_T)]M_\alpha F_{\delta C}\phi'\Big)^2 + r_{22}q_{22}\Big(M_{\delta_E}\big[\omega^2 - F_\eta\big]\Big)^2\Big\} \\
\omega_2^2 &\approx (\omega^2 - F_\eta) + \frac{[1 + (Z_q/V_T)]M_\eta F_\alpha}{\big(\omega^2 - F_\eta\big)} \\
&\quad + \frac{1}{\tilde{k}^2\tilde{\omega}_2^4}\Big[\mathsf{rq}\Big\{r_{11}q_{22}\Big(M_\eta F_{\delta C} + M_{\delta C}\big[\omega^2 - F_\eta\big] \\
&\quad + [1 + (Z_q/V_T)]M_\alpha F_{\delta C}\phi'\Big)^2 + r_{22}q_{22}\Big(M_{\delta_E}\big[\omega^2 - F_\eta\big]\Big)^2\Big\} \\
&\quad + \tilde{\omega}_1^2\tilde{\omega}_2^2\mathsf{rq}\Big\{r_{11}q_{22}\Big(-F_{\delta C}\phi'\Big)^2 + r_{22}q_{22}\Big(M_{\delta_E} - F_{\delta_E}\phi'\Big)^2\Big\}\Big] \\
(2\zeta\omega)_2 &\approx (2\zeta\omega - F_{\dot{\eta}}) + \frac{\{(Z_\eta/V_T) + [1 + (Z_q/V_T)]M_{\dot{\eta}}\}F_\alpha + M_\eta F_q}{\big(\omega^2 - F_\eta\big)} \\
&\quad + \frac{4\tilde{\omega}_1^2 - \tilde{\omega}_2^2}{-2\tilde{k}^2\tilde{\omega}_2^2\widetilde{(2\zeta\omega)}_2}\mathsf{rq}\Big\{r_{11}q_{22}\Big(-F_{\delta C}\phi'\Big)^2 + r_{22}q_{22}\Big(M_{\delta_E} - F_{\delta_E}\phi'\Big)^2\Big\}
\end{aligned}$$

Brief examples of how and why these expressions are useful are discussed subsequently. Aeroelastic mode damping, through feedback, has been increased roughly threefold when Eqs. (27) and (29) are compared. Consider the expression for $(2\zeta\omega)_2$ in Table 2. The first line represents the open-loop contribution, whereas the second line indicates the effect from feedback. One contribution from feedback is represented by the $q_{22}r_{11}$ term. Recall q_{22} penalizes a poor response at the cockpit (q'), and r_{11} penalizes activity of the elevator (δ_E). In other words, larger values for q_{22} and r_{11} imply increased utilization of cockpit pitch rate q' as a feedback signal and canard deflection δ_C as a control input. The $q_{22}r_{11}$ term indicates the cockpit pitch rate to canard feedback path ($q' \rightarrow q_{22}r_{11} \rightarrow \delta_C$) is a key player in the damping increase. Rigorously speaking, q' is not a feedback signal, but its effect through state feedback is.

The associated combination of vehicle parameters $F_{\delta_C}\phi'$ appearing with $q_{22}r_{11}$ tells the physical mechanization for this damping increase. Suppose a disturbance is present in the generalized structural deflection $\dot{\eta}$. Through the mode slope ϕ', this disturbance excites the cockpit response q'. Because of feedback ($q_{22}r_{11}$), the canard (δ_C) is activated. Finally, through the control derivative F_{δ_C}, the initial disturbance in the generalized structural deflection $\dot{\eta}$ is countered. This feedback damping mechanism ($\dot{\eta} \rightarrow \phi' \rightarrow q' \rightarrow q_{22}r_{11} \rightarrow \delta_C \rightarrow F_{\delta_C} \rightarrow \dot{\eta}$) can be emphasized by increases in q_{22} or r_{11}. Likewise, Table 2 indicates the cockpit pitch rate to elevator feedback path ($q' \rightarrow q_{22}r_{22} \rightarrow \delta_E$) is also an important player in the damping augmentation. A simple calculation from Table 2 indicates that the cockpit pitch rate to canard feedback path contributes 67% to damping augmentation, whereas the cockpit pitch rate to elevator feedback path contributes 33%. This 33% contribution is further decomposed into a stabilizing contribution from $F_{\delta_E}\phi'$ and a destabilizing contribution from M_{δ_E}, i.e., damping loss. Information of this type is difficult to come by without the analytical relationships.

Conclusions

A method to uncover functional relationships between design parameters of a contemporary control design technique and the resulting closed-loop properties has been proposed. Efforts concentrated on the LQ state feedback technique applied to an aeroelastic flight control task. For this specific application, simple, roughly accurate analytical expressions for the closed-loop eigenvalues in terms of basic parameters such as stability and control derivatives, structural vibration damping and natural frequency, and cost function weights were generated. These expressions explicitly indicate how the cost weights augment the short period and aeroelastic modes and by what physical mechanization. This type of knowledge is invaluable to the flight control designer and would be more difficult to formulate when obtained from numerical-based sensitivity methods.

References

[1]Bryson, A. E., "New Concepts in Control Theory, 1959–1984," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 4, 1985, pp. 417–425.

[2]Kwakernaak, H., and Sivan, R., *Linear Optimal Control Systems*, Wiley, New York, 1972.

[3]Doyle, J. C., Glover, K., Khargonekar, P. P., and Francis, B. A., "State-Space Solutions to Standard H_2 and H_∞ Control Problems," *Transactions on Automatic Control*, Vol. AC-34, No. 8, 1989, pp. 831–847.

[4]McRuer, D., Ashkenas, I., and Graham, D., *Aircraft Dynamics and Automatic Control*, Princeton Univ. Press, Princeton, NJ, 1973.

[5]Graham, D., "Factors for Cubics and Quartics," *Journal of Guidance, Control, and Dynamics*, Vol. 5, No. 6, 1982, pp. 564–572.

[6]Kwakernaak, H., "Asymptotic Root Loci of Multivariable Linear Optimal Regulators," *Transactions on Automatic Control*, Vol. AC-21, No. 3, 1976, pp. 378–382.

[7]Shaked, U., "The Asymptotic Behavior of the Root-Loci of Multivariable Optimal Regulators," *Transactions on Automatic Control*, Vol. AC-23, No. 3, 1978, pp. 425–430.

[8]Stein, G., "Generalized Quadratic Weights for Asymptotic Regulator Properties," *Transactions on Automatic Control*, Vol. AC-24, No. 4, 1979, pp. 559–566.

[9]Thompson, P. T., Stein, G., and Laub, A. J., "Angles of Multivariable Root Loci," *Transactions on Automatic Control*, Vol. AC-27, No. 6, 1982, pp. 1241–1243.

[10]Pearce, B. F., Johnson, W. A., and Siskind, R. K., "Analytical Study of Approximate Longitudinal Transfer Functions for a Flexible Airframe," Aeronautical Systems Div., Rept. ASD-TDR-62-279, Wright–Patterson AFB, Ohio, June 1962.

[11]Schmidt, D. K., and Newman, B., "Modeling, Model Simplification and Stability Robustness with Aeroelastic Vehicles," *Proceedings of the AIAA Guidance, Navigation, and Control Conference* (Minneapolis, MN), AIAA, Washington, DC, 1988, pp. 210–221.

[12]Newman, B., and Schmidt, D. K., "Numerical and Literal Aeroelastic-Vehicle-Model Reduction for Feedback Control Synthesis," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 5, 1991, pp. 943–953.

[13]Newman, B., "Aerospace Vehicle Model Simplification for Feedback Control," Ph.D. Dissertation, School of Aeronautics and Astronautics, Purdue Univ., West Lafayette, IN, Aug. 1992.

[14]Livneh, R., and Schmidt, D. K., "New Literal Approximations for the Longitudinal Dynamic Characteristics of Flexible Flight Vehicles," *Proceedings of the AIAA Guidance, Navigation, and Control Conference* (Hilton Head, SC), AIAA, Washington, DC, 1992, pp. 536–545.

[15]Livneh, R., "Improved Literal Approximations for the Lateral-Directional Dynamics of Rigid Aircraft," *Proceedings of the AIAA Guidance, Navigation, and Control Conference* (Baltimore, MD), AIAA, Washington, DC, 1995, pp. 1197–1207.

[16]McRuer, D. T., Meyers, T. T., and Thompson, P. M., "Literal Singular-Value-Based Flight Control System Design Techniques," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 6, 1989, pp. 913–919.

[17]Newman, B., and Schmidt, D. K., "Aeroelastic Vehicle Multivariable Control Synthesis with an Analytical Robustness Evaluation," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 6, 1994, pp. 1145–1153.

[18]Boeing Commercial Airplane Group, "High-Speed Civil Transport Study," NASA-CR-4233, Sept. 1989.

[19]Douglas Aircraft Co., "Study of High-Speed Civil Transports," NASA-CR-4235, Dec. 1989.

[20]Newman, B., and Buttrill, C., "Conventional Flight Control for an Aeroelastic, Relaxed Static Stability High-Speed Transport," *Proceedings of the AIAA Guidance, Navigation, and Control Conference* (Baltimore, MD), AIAA, Washington, DC, 1995, pp. 717–726.

[21]Waszak, M. R., and Schmidt, D. K., "Flight Dynamics of Aeroelastic Vehicles," *Journal of Aircraft*, Vol. 25, No. 6, 1988, pp. 563–571.